Maximum Matching Width:
new characterizations and a fast algorithm for dominating set

정지수 (KAIST)

joint work with Sigve Hortemo Sæther and Jan Arne Telle
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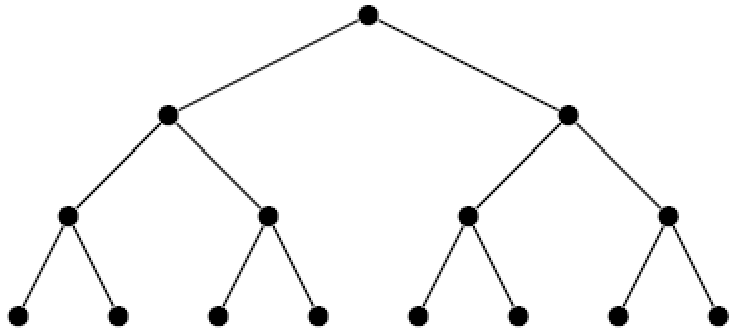
Graph width parameters

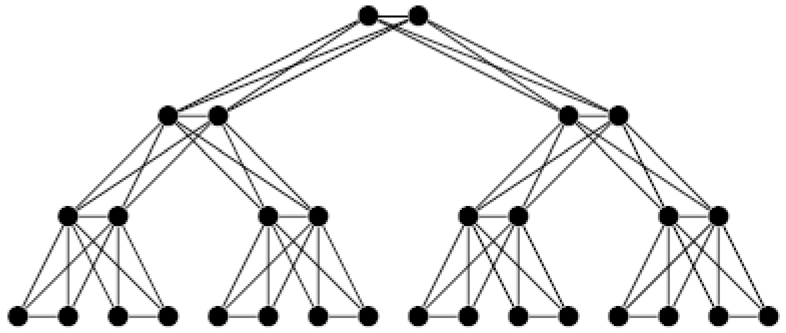
- tree-width (Halin 1976, Robertson and Seymour 1984)
- branch-width (Robertson and Seymour 1991)
- carving-width (Seymour and Thomas 1994)
- clique-width (Courcelle and Olariu 2000)
- rank-width (Oum and Seymour 2006)
- boolean-width (Bui-Xuan, Telle, Vatshelle 2011)
- maximum matching-width (Vatshelle 2012)

Tree-width

- tree-width (Halin 1976, Robertson and Seymour 1984)
  A measure of how "tree-like" the graph is.



tree



tree-like

Figures from http://fptschool.mimuw.edu.pl/slides/lec6.pdf

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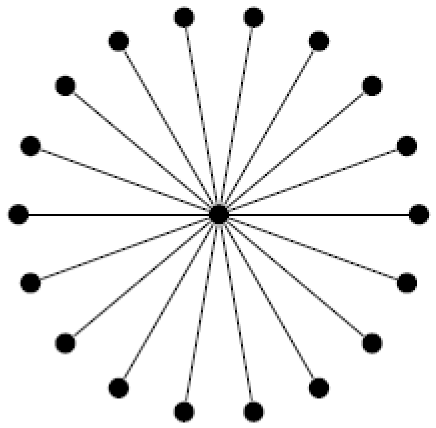
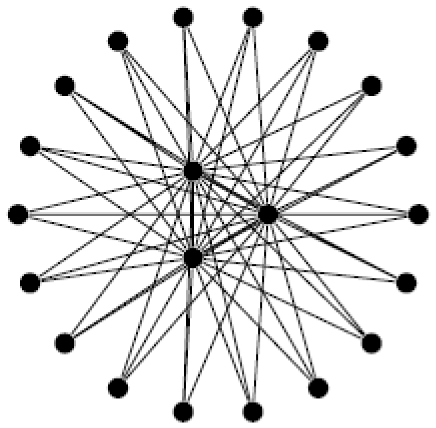


tree

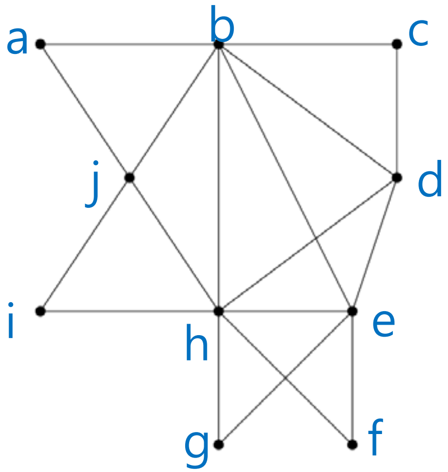


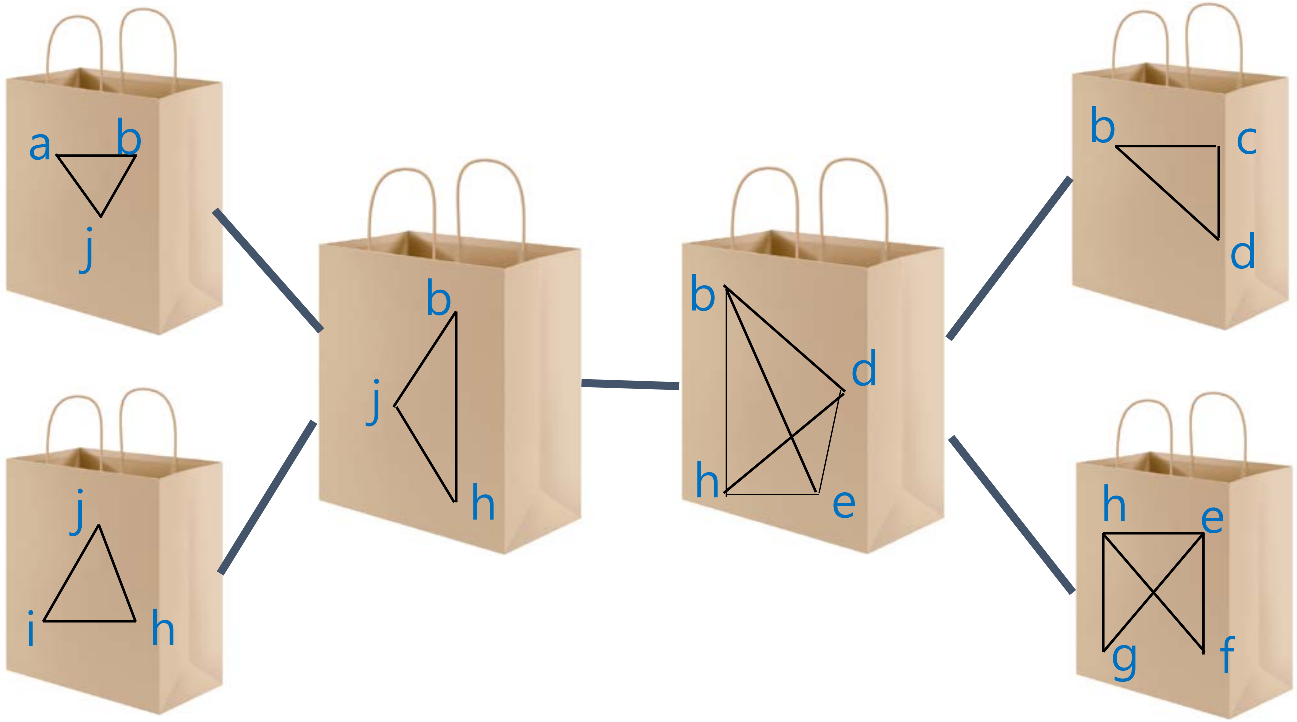
tree-like

Figures from http://fptschool.mimuw.edu.pl/slides/lec6.pdf

A *tree-decomposition* of a graph G is a pair $(T, \{X_t\}_{t\in V(T)})$ consisting of a tree T and a family $\{X_t\}_{t\in V(T)}$ of subsets X_t of $V(G)$, called *bags*, satisfying the following three conditions:

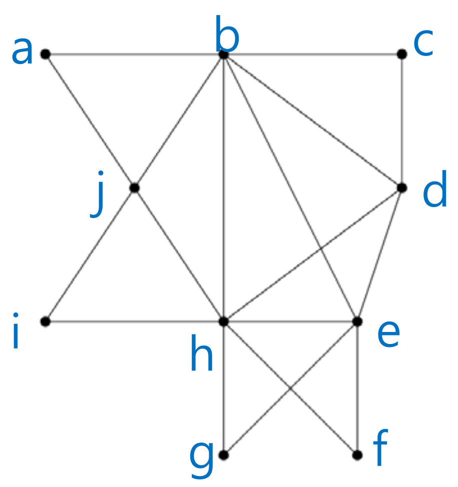
1. each vertex of G is in at least one bag,
2. for each edge uv of G, there exists a bag that contains both u and v,
3. if X_i and X_j both contain a vertex v, then all bags X_k in the path between X_i and X_j contain v as well.

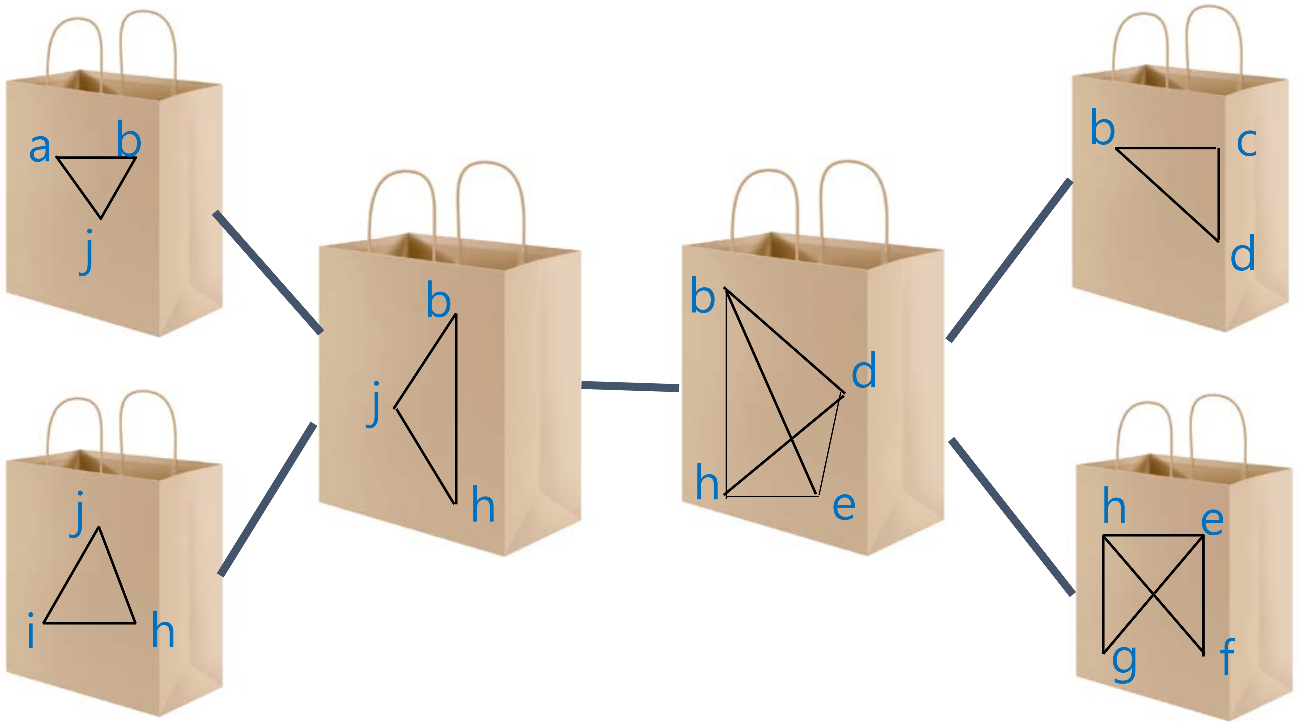




The *width* of a tree-decomposition $(T, \{X_t\}_{t \in V(T)})$ is $\max|X_t| - 1$.

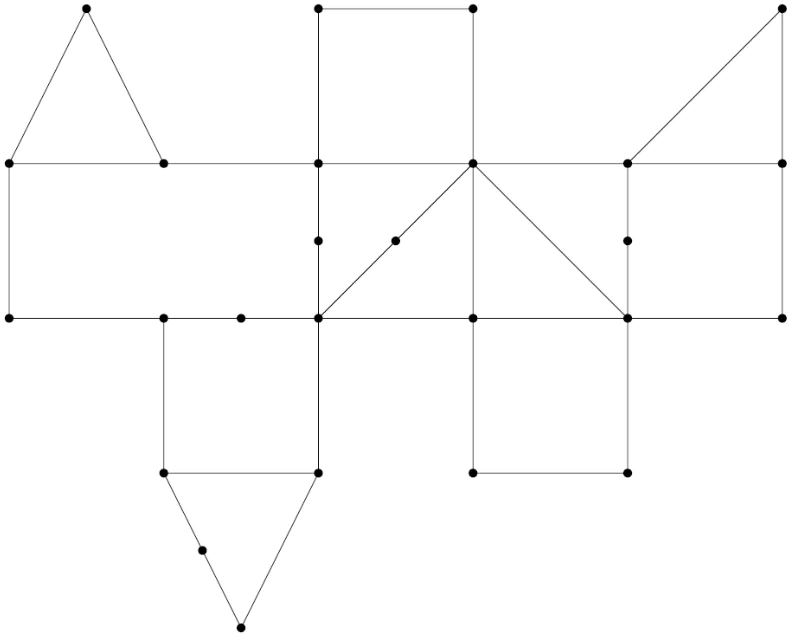
The *tree-width* of a graph G, denoted by $tw(G)$, is the minimum width over all possible tree-decompositions of G.

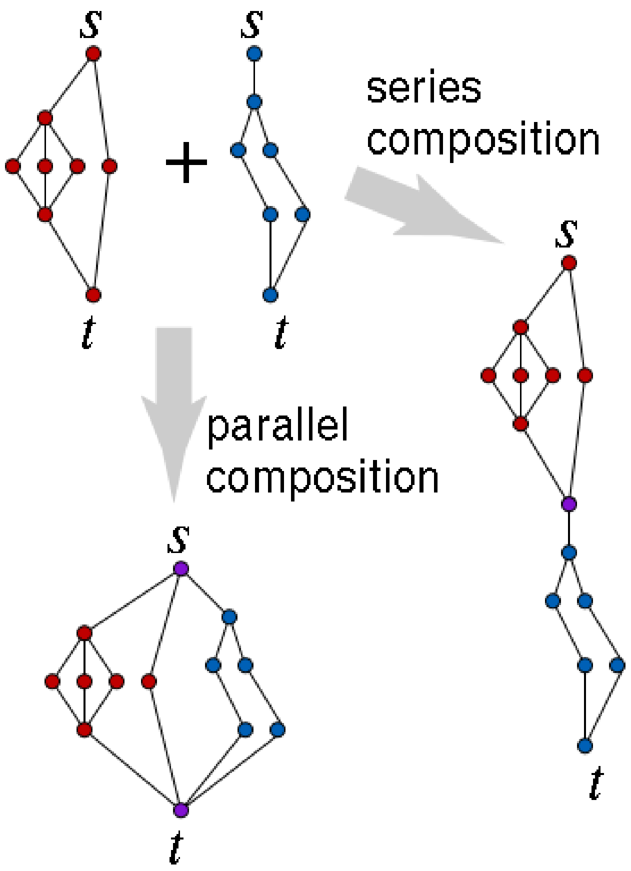




Examples

- tree-width $\leq 1 \Leftrightarrow$ a forest $\Leftrightarrow$ no cycle
- tree-width $\leq 2 \Leftrightarrow$ a series-parallel graph
  $\Leftrightarrow$ no K_4 minor



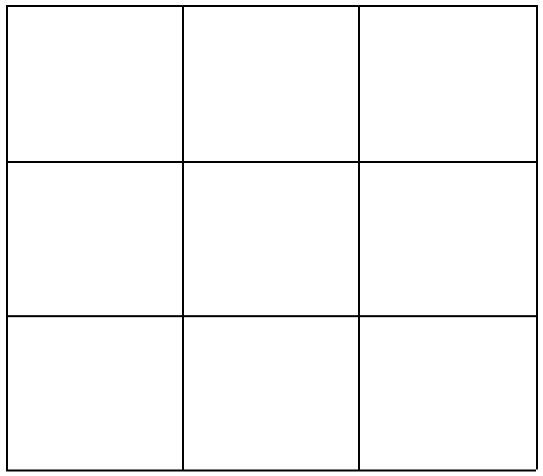


Figures from wikipedia

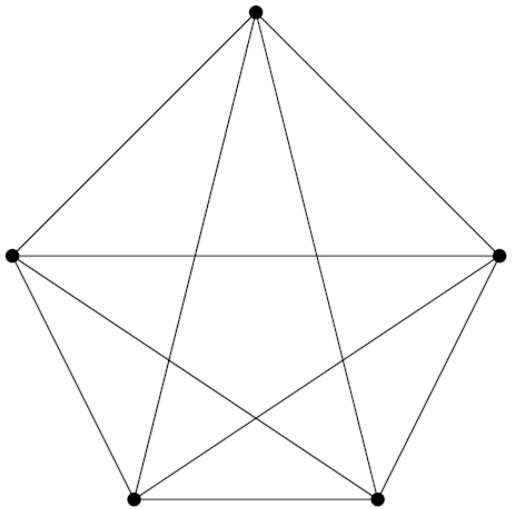
Examples

- tree-width $\leq 1 \Leftrightarrow$ a forest $\Leftrightarrow$ no cycle
- tree-width $\leq 2 \Leftrightarrow$ a series-parallel graph

  $\Leftrightarrow$ no K_4 minor
- The tree-width of a $k \times k$ grid is k.
- The tree-width of K_n is $n - 1$.



4×4 grid



K_5

by Fedor V. Fomin
(http://fptschool.mimuw.edu.pl/slides/lec6.pdf)

Why treewidth?

Very general idea in science: large structures can be understood by breaking them into small pieces

In Computer Science: divide and conquer; dynamic programming

In Graph Algorithms: Exploiting small separators

Courcelle's theorem

Theorem (Courcelle 1990)

Let P be a property that can be expressed in MSO_2 logic.
If G is a graph of bounded tree-width, then there exists a linear-time algorithm that tests whether G has property P.

e.g.
3-COLORING, HAMILTONICITY, SUBGRAPH ISOMORPHISM, MINOR TEST, VERTEX COVER, DOMINATING SET, INDEPENDENT SET, STEINER TREE, FEEDBACK VERTEX SET

Excluded Grid Theorem

The tree-width of a $k \times k$ grid is k.
If a graph contains a large grid as a minor, then its tree-width is also large.

Theorem (Robertson, Seymour, Thomas 1994)

If the tree-width of a graph G is at least 20^{2k^5},
then G has a $k \times k$ grid as a minor.

Theorem (Robertson, Seymour, Thomas 1994)

If the tree-width of a planar graph G is at least $6k - 4$,
then G has a $k \times k$ grid as a minor.

Graph width parameters

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- clique-width (Courcelle and Olariu 2000)
- rank-width (Oum and Seymour 2006)
- boolean-width (Bui-Xuan, Telle, Vatshelle 2011)
- maximum matching-width (Vatshelle 2012)

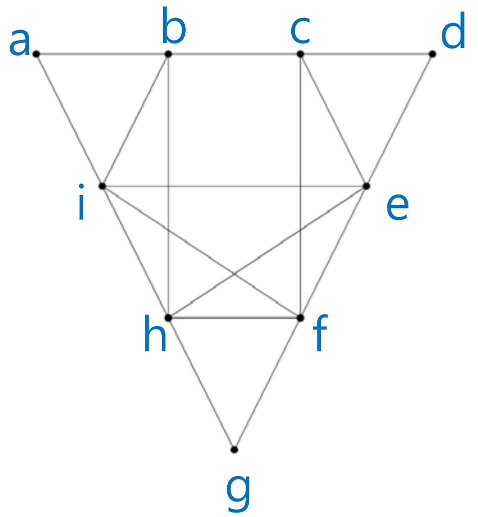
A *branch-decomposition* (T, L) over the vertices of a graph G consists of

a tree T where all internal vertices have degree 3 and

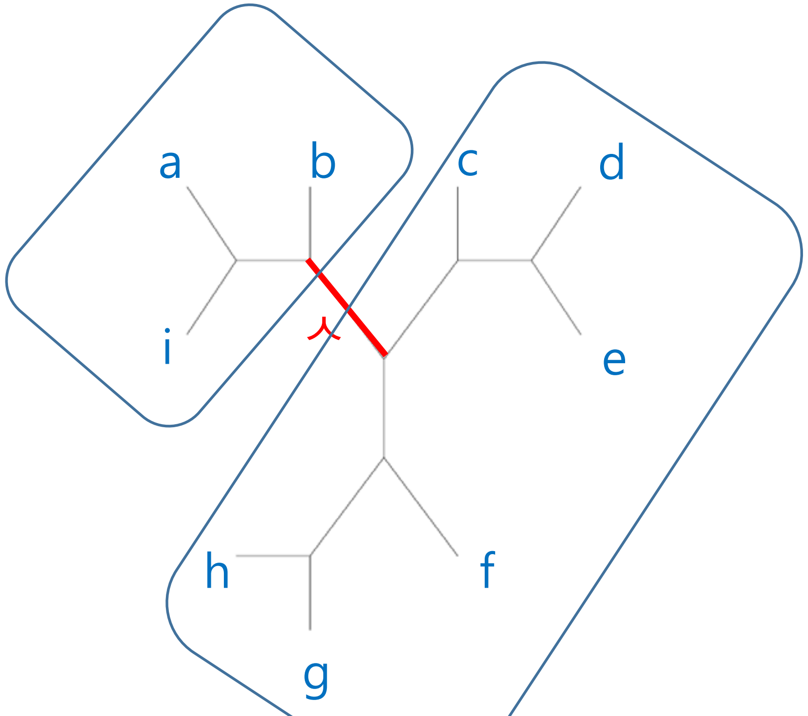
a bijective function L from the leaves of T to the vertices of G.

The value of an edge ㅅ of T is

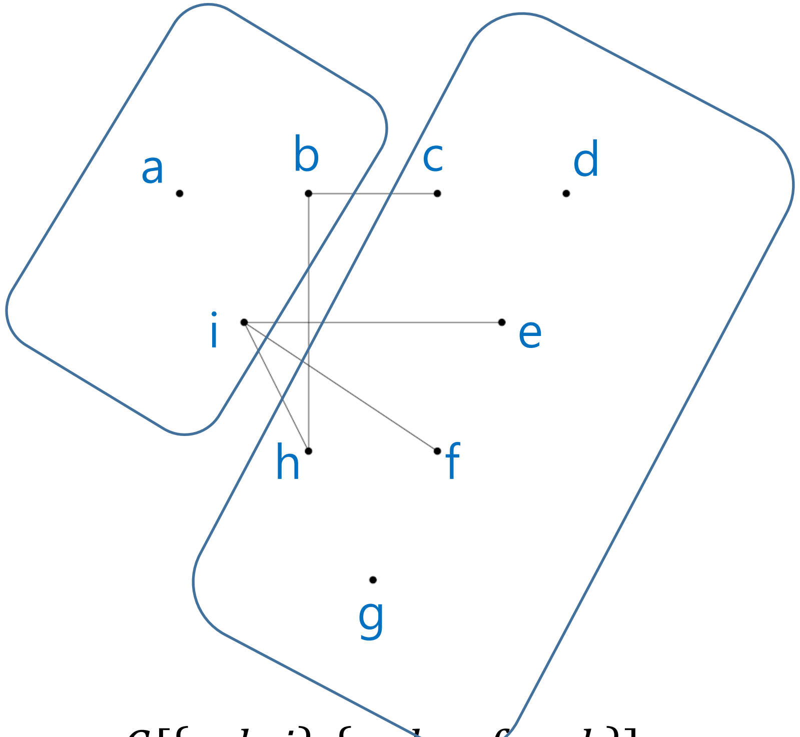
the size of the maximum matching of $G[\{a, b, i\}, \{c, d, e, f, g, h\}]$.



G



(T, L)



$G[\{a, b, i\}, \{c, d, e, f, g, h\}]$

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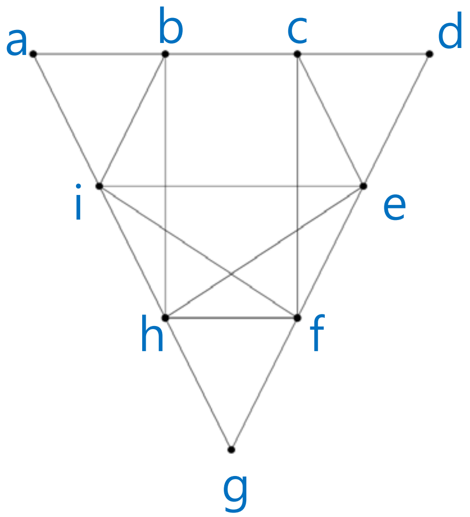

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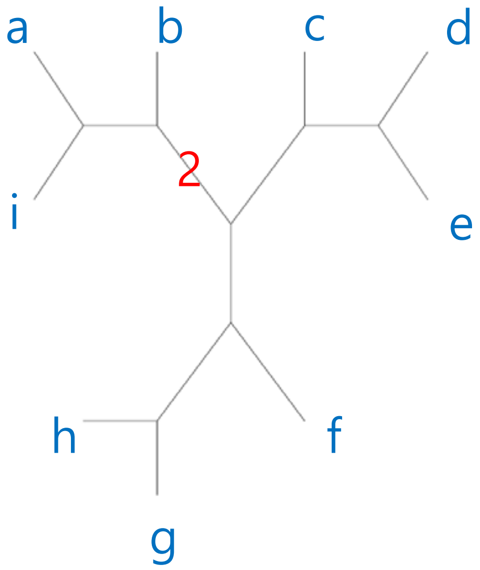

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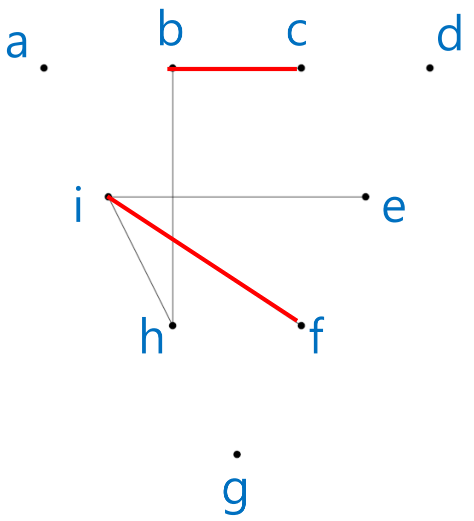


$G[\{a, b, i\}, \{c, d, e, f, g, h\}]$

A *branch-decomposition* (T, L) over the vertices of a graph G consists of

a tree T where all internal vertices have degree 3 and

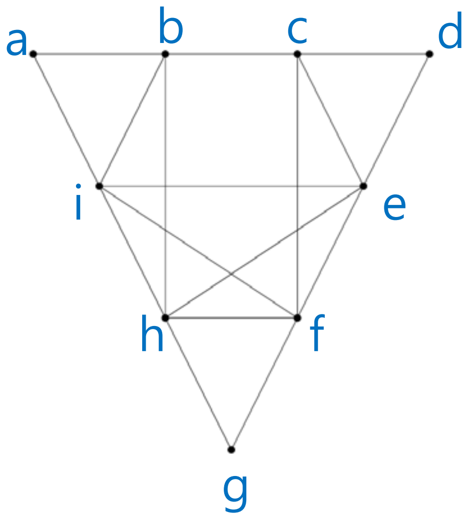
a bijective function L from the leaves of T to the vertices of G.

The width of a branch-decomposition (T, L) is

the maximum value among all edges.

The maximum matching-width (mm-width, $mmw(G)$) of a graph G is

the minimum width over all possible branch-decompositions over $V(G)$.



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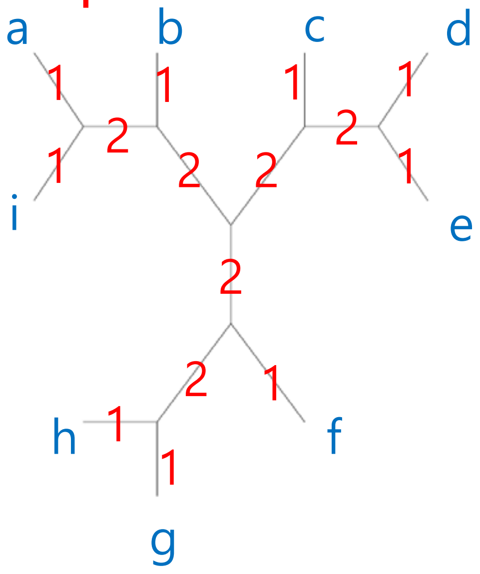


(T, L)

The width of (T, L) is 2

Why mm-width?

Theorem (Vatshelle 2012)

For every graph G,

$$mmw(G) \leq tw(G) + 1 \leq 3\, mmw(G).$$

A graph G has bounded tree-width if and only if G has bounded mm-width.

Why mm-width?

We want to solve Graph Problems efficiently.

A Dominating Set of a graph G is a set D of vertices such that $N(D) \cup D = V(G)$.

What is the minimum size of a dominating set of G?

Why mm-width?

Using tree-width

Theorem (van Rooij, Bodlaender, Rossmanith 2009)

Minimum Dominating Set Problem can be solved in time $O^*(3^t)$ when a graph and its tree-decomposition of width t is given.

Theorem (Lokshtanov, Marx, Saurabh 2011)

Minimum Dominating Set Problem cannot be solved in time $O^*((3-\varepsilon)^t)$ where t is the tree-width of the given graph.

Why mm-width?

Using mm-width

Theorem (J., Sæther, Telle 2015+)

Minimum Dominating Set Problem can be solved in time $O^*(8^m)$ when a graph and its branch-decomposition of mm-width m is given.

Why mm-width?

Using tree-width: $O^*(3^t)$

Using mm-width: $O^*(8^m)$

Our algorithm is faster when $8^m < 3^t$, that is,

$$1.893\, mmw(G) < tw(G).$$

Note that for every graph G,

$$mmw(G) \leq tw(G) + 1 \leq 3\, mmw(G).$$

Why mm-width?

Using mm-width

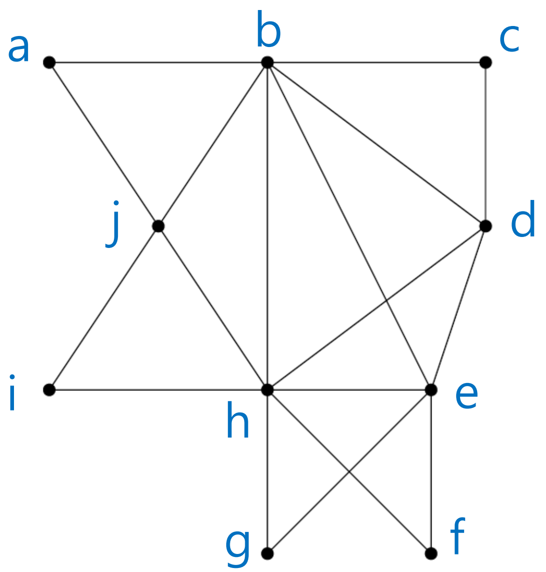
Theorem (J., Sæther, Telle 2015+)

Minimum Dominating Set Problem can be solved in time $O^*(8^m)$ when a graph and its branch-decomposition of mm-width m is given.

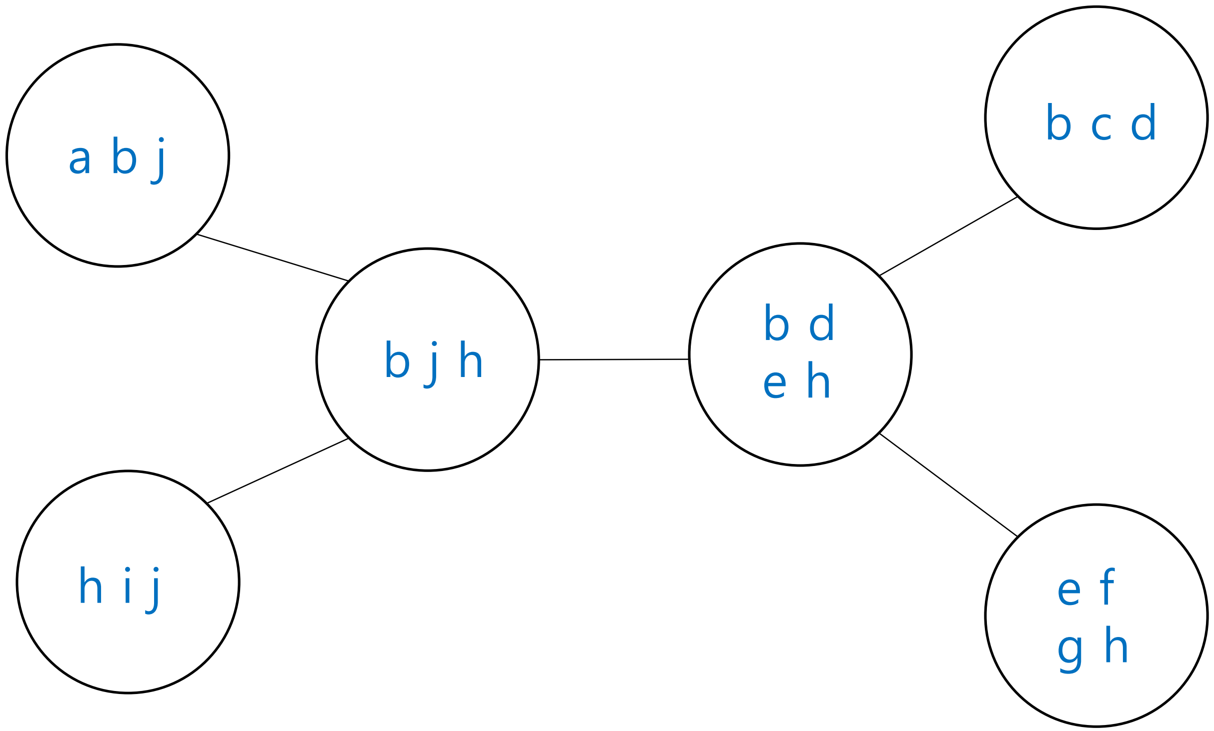
Proof ideas

1. New characterization of graphs of mm-width at most k
2. Fast Subset Convolution

New characterization (tree-width)



graph



tree-decomposition

New characterization (tree-width)

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \ldots, v_n$ has tree-width at most k *if and only if* there are subtrees $T_1, T_2, \ldots, T_n$ of a tree T where all internal vertices have degree 3

such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex of T in common,

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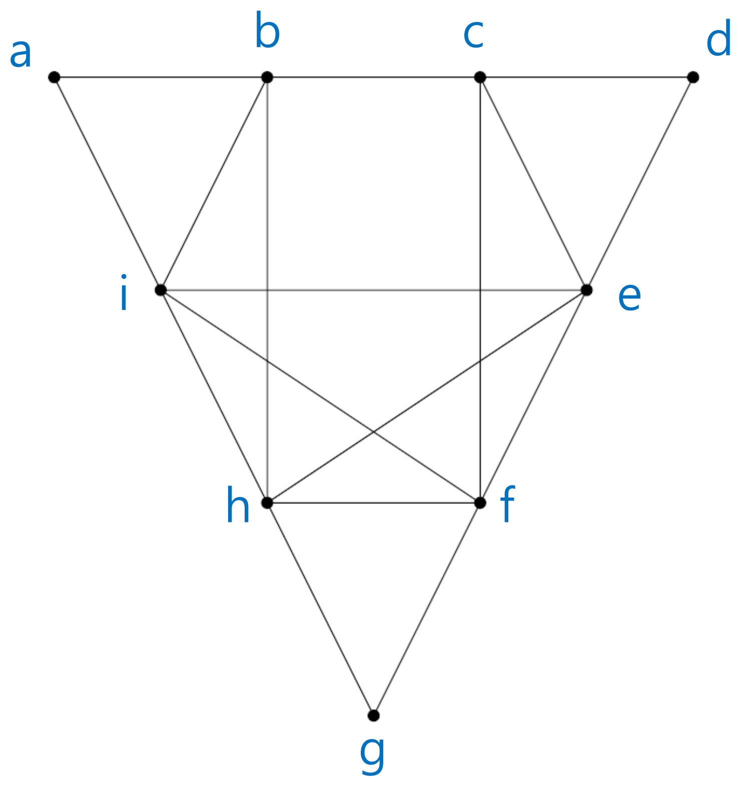
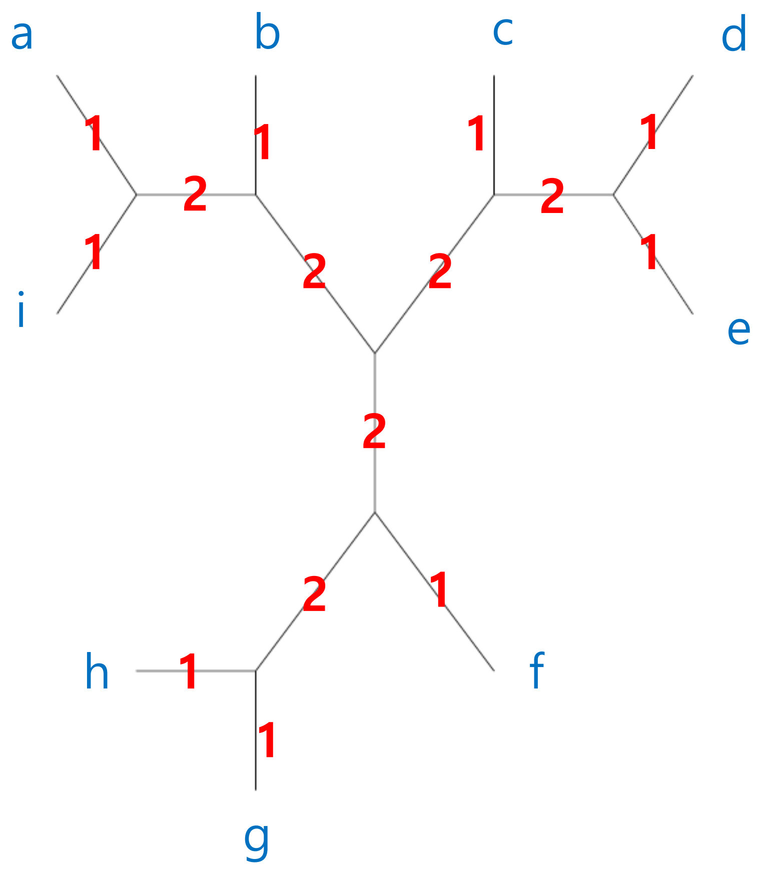
New characterization (mm-width)





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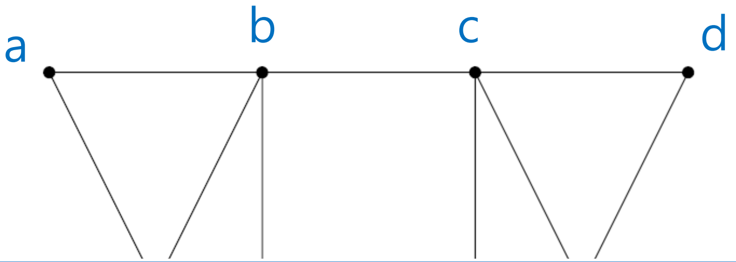
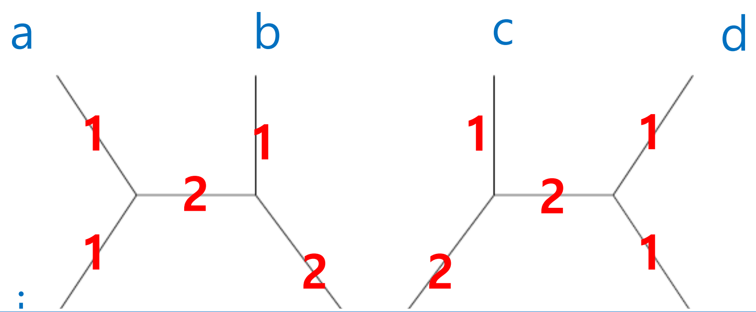
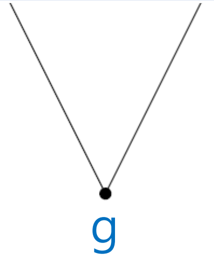
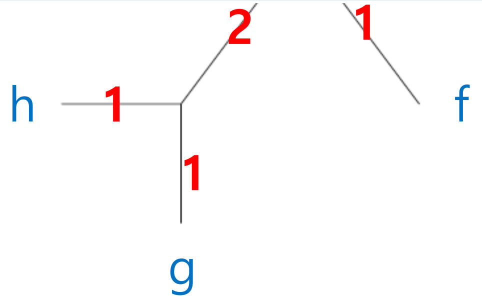




Theorem (König 1931)

For every bipartite graph G,
the size of a maximum matching is equal to the size of a minimum vertex cover.





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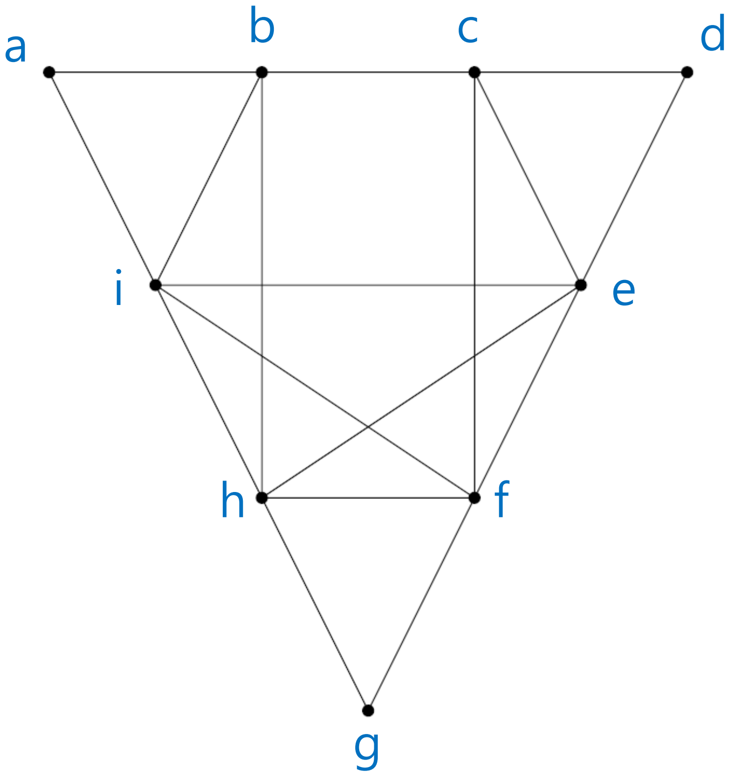

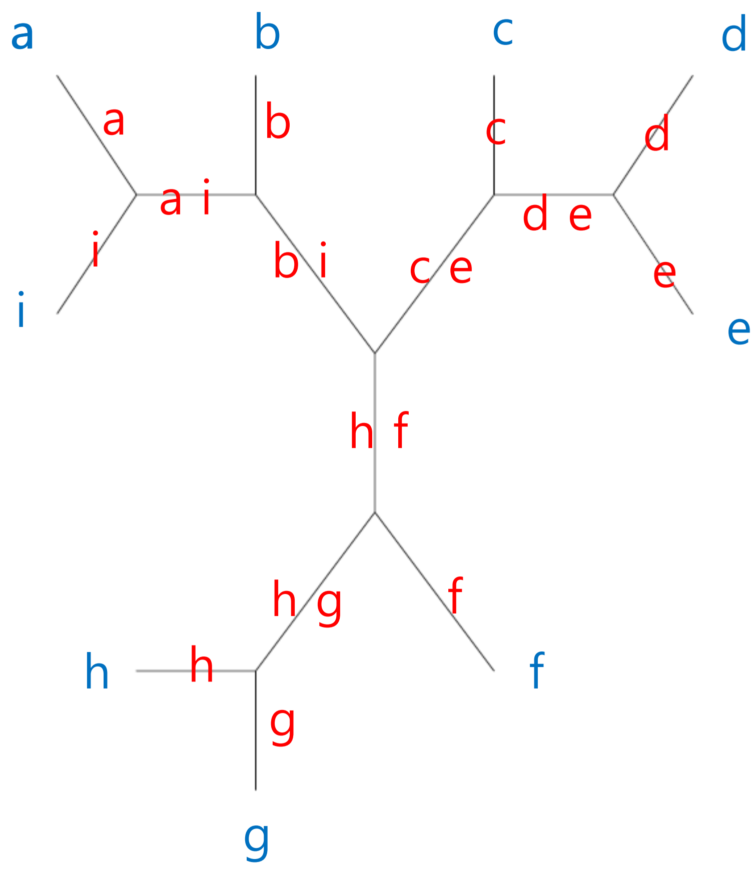

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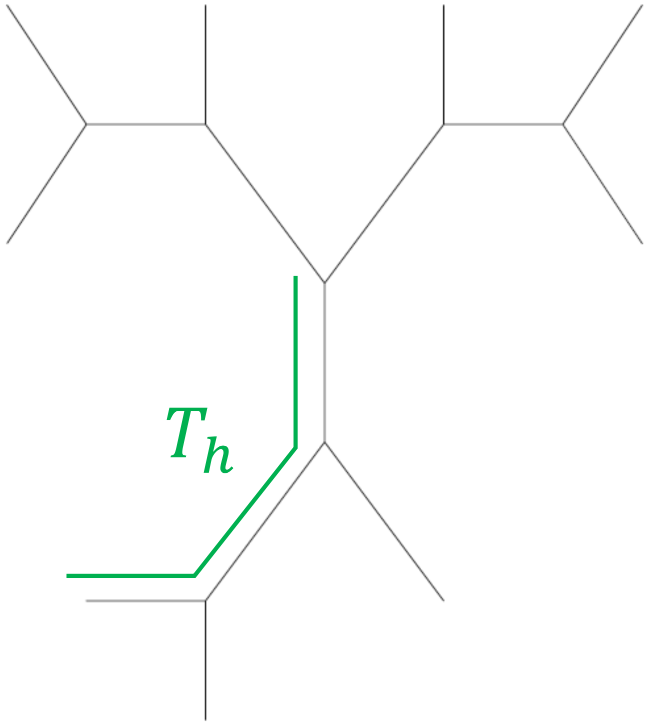

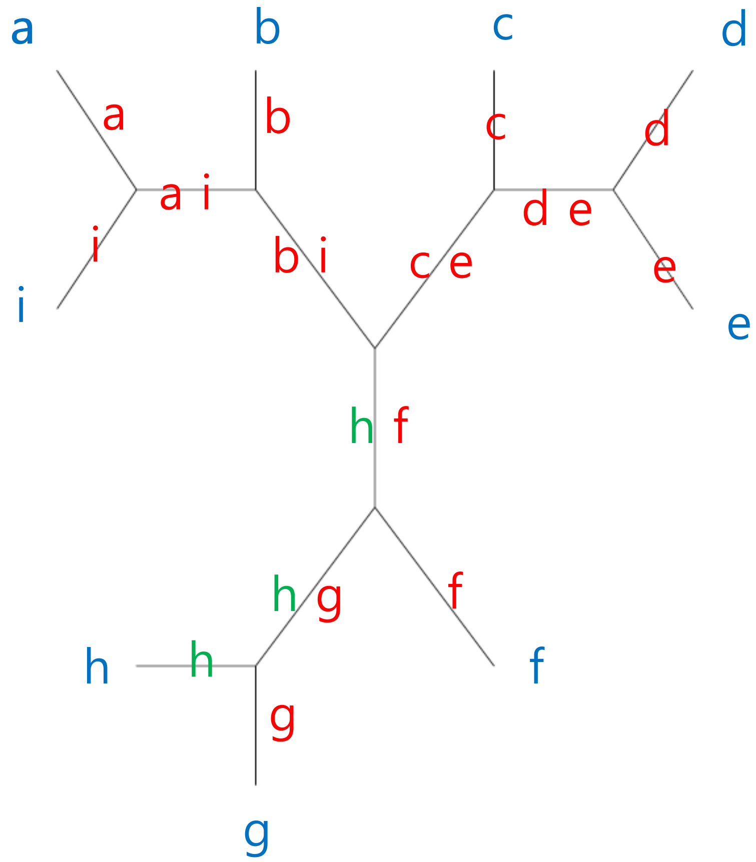

New characterization (mm-width)

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such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex of T in common,

2) for each edge of T, there are at most k subtrees containing it.

Theorem (J., Sæther, Telle 2015+)

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2) for each edge of T, there are at most k subtrees containing it.

New characterization

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \ldots, v_n$ has tree-width (mm-width) at most k *if and only if* there are subtrees $T_1, T_2, \ldots, T_n$ of a tree T where all internal vertices have degree 3 such that 1) if $v_i v_j \in E(G)$, then T_i and T_j have at least one vertex of T in common,

2) for each vertex (edge) of T, there are at most $k-1$ (at most k) subtrees containing it.

Thank you

New characterization (branch-width)

For any $k \geq 2$, a graph G on vertices $v_1, v_2, \ldots, v_n$ has branch-width at most k *if and only if* there are a tree T of max degree at most 3 and subtrees $T_1, T_2, \ldots, T_n$ such that 1) if $v_i v_j \in E(G)$ then T_i and T_j have at least one edge of T in common, 2) for each edge of T, there are at most k subtrees using it.